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ON THE STATE OF STRESS AND STRAIN IN A FINITE CYLINDER SUBJECTED TO DYNAMIC LOADS

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G. A. BRUSILOVSKAIA and L. V. ERSHOV
(Moscow)

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A solution is presented of the dynamical axisymmetric problem of elasticity theory for a cylinder of arbitrary length with given displacements on its curved and planar surfaces. The initial non-self-adjoint equations are converted into equivalent first order equations for an extended eigenvector by introducing certain auxiliary functions. Arbitrary displacements given on the flat endface of the cylinder are expanded in series of eigensolutions of the problem by using these eigenvectors. Final formulas are obtained for the expansion coefficients. As a particular case, the solution of the statics problem of a cylinder [1] follows for $\omega \to 0$. An analogous problem has been examined in [2] where it was reduced to solving an infinite system of equations. The numerical method for solving problems of such a class has been elucidated in [3].

1. Let us proceed from the differential equations in displacements

$$v_1^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)+v_2^2\frac{\partial^2 u}{\partial z^2}+ \qquad (1.1)$$

$$(v_2^2-v_1^2)\left(\frac{\partial^2 w}{\partial z\partial r}+\frac{1}{r}\frac{\partial w}{\partial z}\right)-\frac{\partial^2 u}{\partial t^2}=0$$

$$v_2^2\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}-\frac{w}{r^2}\right)+v_1^2\frac{\partial^2 w}{\partial z^2}+(v_2^2-v_1^2)\frac{\partial^2 u}{\partial z\,\partial r}-\frac{\partial^2 w}{\partial t^2}=0$$

$$v_1^2=\frac{\mu}{\rho},\qquad v_2^2=\frac{\lambda+2\mu}{\rho}$$

Let the boundary conditions be

$$u(r,z,t)\big|_{r=a}=\varphi_1(z)\,e^{i\omega t} \qquad (1.2)$$
$$w(r,z,t)\big|_{r=a}=\varphi_2(z)\,e^{i\omega t}$$

$$u(r,z,t)\big|_{z=0}=g_1(r)\,e^{i\omega t},\qquad u(r,z,t)\big|_{z=l}=f_1(r)\,e^{i\omega t} \qquad (1.3)$$
$$w(r,z,t)\big|_{z=0}=g_2(r)\,e^{i\omega t},\qquad w(r,z,t)\big|_{z=l}=f_2(r)\,e^{i\omega t}$$

Here λ, μ are Lamé constants, ρ is the material density, u, w are the longitudinal and radial displacements, respectively, l is the length, a is the radius of the cylinder, and ω is the frequency of the forcing term. It is assumed that the frequency of the forcing term does not coincide with any natural frequency of the cyclinder, and the following relationships

$$\varphi_1(z)\big|_{z=0}=g_1(r)\big|_{r=a},\qquad \varphi_2(z)\big|_{z=0}=g_2(r)\big|_{r=a}$$
$$\varphi_1(z)\big|_{z=l}=f_1(r)\big|_{r=a},\qquad \varphi_2(z)\big|_{z=l}=f_2(r)\big|_{r=a}$$

are satisfied under the conditions (1.2), (1.3). Let us seek the displacements u and w as the sum of solutions of (1.1) for an infinite cylinder with the known displacements (1.2) on the side surface (see Sect. 2) and for a semi-infinite cylinder with zero displacements on the side surface but the known displacements (1.3) on its flat endface (see Sect. 3). Superposition of the solutions permits considering the general solution for an elastic cylinder of arbitrary length with given displacements of the form (1.2) and (1.3) on its side surfaces. Having determined the displacements, the strain can be found by means of known formulas, and the stress on the basis of the elastic strain law. Their expressions are omitted in the text.

2. We take the solution of (1.1) for an infinite cylinder with the boundary conditions (1.2) as

$$u_1(r,z,t)=u_1(r)\sin(\beta_n z)\,e^{i\omega t} \qquad (2.1)$$
$$w_1(r,z,t)=w_1(r)\cos(\beta_n z)\,e^{i\omega t},\quad \beta_n=n\pi/l$$

Substituting (2.1) into (1.1), we obtain a system of ordinary differential equations for the functions $u_1(r)$ and $w_1(r)$

$$v_1^2\left(u_1''+\frac{1}{r}u_1'\right)-\beta_n^2v_2^2u_1-\beta_n(v_2^2-v_1^2)\left(w_1'+\frac{1}{r}w_1\right)+ \qquad (2.2)$$
$$w^2u_1=0$$
$$v_2^2\left(w_1''+\frac{1}{r}w_1'-\frac{w_1}{r^2}\right)-\beta_n^2v_1^2w_1+\beta_n(v_2^2-v_1^2)u_1'+\omega^2w_1=0$$

Hence, we write the solution of the system (2.2) for each harmonic $(n=1,2,3\ldots)$ as follows:

$$u_{1n}(r)=A_n\beta_nI_0(\gamma_{1n}r)+B_n\frac{\beta_nv_1^2}{a\omega^2(k-1)}\left[\beta_nI_0(\gamma_{1n}r)-\gamma_{2n}I_0(\gamma_{2n}r)\right]$$
$$w_{1n}(r)=A_n\gamma_{1n}I_1(\gamma_{1n}r)+B_n\frac{\beta_nv_1^2}{a\omega^2(k-1)}\left[\gamma_{1n}I_1(\gamma_{1n}r)-\beta_nI_1(\gamma_{2n}r)\right]$$
$$\gamma_{1n}^2=\beta_n^2-\frac{\omega^2}{v_2^2},\qquad \gamma_{2n}^2=\beta_n^2-\frac{\omega^2}{v_1^2},\qquad k=\frac{v_1^2}{v_2^2}$$

where A_n, B_n are continuous in ω, and $I_0(x)$, $I_1(x)$ are modified Bessel functions.

Furthermore, assuming $l\omega < \pi v_1$, we determine the constants A_n and B_n from the boundary conditions (1.2).

The solution of (1.1) corresponding to $\beta_n = 0$, can be obtained by the direct solution of (2.2), where the constants of integration are determined analogously to A_n and B_n. The sum of the solutions obtained yields the desired solution about the displacements of an infinite cylinder with the boundary conditions (1.2).

Let us note that the solution of (1.1) is conveniently taken in the form (2.1) under the condition that $\varphi_1(z)$ is an odd and $\varphi_2(z)$ an even function. In order to be able to expand any boundary values of the displacements, it is necessary to add equivalent relationships obtained for mutual commutation of the sines and cosines to (2.1).

3. Let us consider a semi-infinite cylinder on whose curved surfaces the displacements equal zero, while the first two conditions (1.3) are given on the flat endface.

We seek the solution of the initial system of equations as

$$u_2(r, z, t) = u_2(r)\, e^{-\alpha z/a}\, e^{i\omega t} \qquad (3.1)$$
$$w_2(r, z, t) = w_2(r)\, e^{-\alpha z/a}\, e^{i\omega t}$$

Substituting (3.1) into (1.1) and solving the system analogous to (2.2) for the functions $u_2(r)$ and $w_2(r)$, we have

$$u_2(r) = C\alpha J_0(\delta_1 r) + D\frac{\alpha v_1^2}{a\omega^2(k-1)}\left[\frac{\alpha}{a}J_0(\delta_1 r) - \delta_2 J_0(\delta_2 r)\right] \qquad (3.2)$$
$$w_2(r) = C\delta_1 a J_1(\delta_1 r) + D\frac{\alpha v_1^2}{a\omega^2(k-1)}\left[\delta_1 J_1(\delta_1 r) - \frac{\alpha}{a}J_1(\delta_2 r)\right]$$
$$\delta_1^2 = \frac{\alpha^2}{a^2} + \frac{\omega^2}{v_2^2}, \qquad \delta_2^2 = \frac{\alpha^2}{a^2} + \frac{\omega^2}{v_1^2}$$

The parameter α is an eigenvalue and is determined from the homogeneous boundary conditions on the curved surface, which can be written as follows: $u_2(a, z, t) \equiv 0$, $w_2(a, z, t) \equiv 0$ or taking account of (3.1) and (3.2)

$$C\alpha J_0(\delta_1 a) + D\frac{\alpha v_1^2}{a\omega^2(k-1)}\left[\frac{\alpha}{a}J_0(\delta_1 a) - \delta_2 J_0(\delta_2 a)\right] = 0 \qquad (3.3)$$
$$C\delta_1 a J_1(\delta_1 a) + D\frac{\alpha v_1^2}{a\omega^2(k-1)}\left[\delta_1 J_1(\delta_1 a) - \frac{\alpha}{a}J_1(\delta_2 a)\right] = 0$$

The characteristic equation to determine the eigenvalues α hence follows

$$\frac{\alpha^2}{a^2}J_0(\delta_1 a)J_1(\delta_2 a) - \delta_1\delta_2 J_0(\delta_2 a)J_1(\delta_1 a) = 0 \quad (\text{for } \omega \neq 0) \qquad (3.4)$$

$$\alpha J_0^2(\alpha) - \frac{2}{1-k}J_0(\alpha)J_1(\alpha) + \alpha J_1^2(\alpha) = 0 \quad (\text{for } \omega = 0) \qquad (3.5)$$

Equation (3.5) agrees with the characteristic equation in [1], which was obtained in the statics problem for an elastic cylinder. The transcendental equation (3.4) containing the parameter α in the arguments of the Bessel functions as well as outside has an infinite denumerable set of roots α_n $(n = 1,2,3,\ldots)$.

It should be noted that $\alpha = 0$ is not a root of (3.4) for a cylinder whose parameters satisfy the inequality $2a \leqslant l$.

Investigations carried out show that in addition to

$$\alpha_{1,2} \approx \pm \sqrt{4k - \frac{\omega^2 a^2}{v_2^2}} \quad \text{(first approximation)}$$

$$\alpha_{3,4} = \pm \frac{\omega a}{v_1} i$$

all the roots of the transcendental equation (3.4) are complex-conjugates grouped in the quadrant

$$\alpha = \pm c_1 \pm d_1 i$$

Let us note that the eigenvalue $\alpha = \pm (\omega a / v_1)\, i$ corresponds to the trivial solution. Only the roots with positive real parts are of interest since they assure damping with increasing z. For each such α_n we have from the second equation in (3.3)

$$\frac{C_n}{D_n} = \frac{\alpha_n v_1^2}{a^2\omega^2(1-k)} \left[1 - \frac{\alpha_n}{a\delta_{1n}} \frac{J_1(\delta_{2n}a)}{J_1(\delta_{1n}a)}\right] \tag{3.6}$$

and the solutions of (3.2) become

$$u_{2n}(r) = \frac{\alpha_n v_1^2}{a\omega^2(1-k)} \left[-\frac{\alpha_n^2 J_1(\delta_{2n}a)}{a^2\delta_{1n} J_1(\delta_{1n}a)} J_0(\delta_{1n}r) + \delta_{2n} J_0(\delta_{2n}r)\right] \tag{3.7}$$

$$w_{2n}(r) = \frac{\alpha_n v_1^2}{a\omega^2(1-k)} \left[-\frac{\alpha_n J_1(\delta_{2n}a)}{a J_1(\delta_{1n}a)} J_1(\delta_{1n}r) + \frac{\alpha_n}{a} J_1(\delta_{2n}r)\right]$$

Summing over all values of α_n, we represent the solution of (3.1) as infinite series containing the unknown constants d_n

$$u_2(r, z, t) = \sum_{n=1}^{\infty} d_n u_{2n}(r) \exp\left(-\frac{\alpha_n z}{a}\right) \exp(i\omega t) \tag{3.8}$$

$$w_2(r, z, t) = \sum_{n=1}^{\infty} d_n w_{2n}(r) \exp\left(-\frac{\alpha_n z}{a}\right) \exp(i\omega t)$$

Substituting (3.8) in the first two conditions (1.3), we obtain

$$\sum_{n=1}^{\infty} d_n u_{2n}(r) = g_1(r), \qquad \sum_{n=1}^{\infty} d_n w_{2n}(r) = g_2(r) \tag{3.9}$$

To determine the unknown constants d_n, let us write the initial system (1.1) taking account of (3.1) in the matrix form

$$[r\xi'(r)]' = \alpha L_1 \xi'(r) + \alpha^2 L_2 \xi(r) + \alpha L_3 \xi(r) + L_4 \xi(r) \tag{3.10}$$

Here

$$\xi(r) = \begin{Vmatrix} u_2(r) \\ w_2(r) \end{Vmatrix}, \qquad L_1 = \begin{Vmatrix} 0 & \left(\frac{1}{k} - 1\right)\frac{r}{a} \\ (1-k)\frac{r}{a} & 0 \end{Vmatrix}$$

$$L_2 = \begin{Vmatrix} -\frac{1}{k}\frac{r}{a^2} & 0 \\ 0 & -\frac{kr}{a^2} \end{Vmatrix}, \qquad L_3 = \begin{Vmatrix} 0 & \frac{1}{a}\left(\frac{1}{k} - 1\right) \\ 0 & 0 \end{Vmatrix}, \qquad L_4 = \begin{Vmatrix} -\frac{\omega^2}{v_1^2} r & 0 \\ 0 & \frac{1}{r} - \frac{\omega^2}{v_2^2} r \end{Vmatrix}$$

The boundary conditions for $\xi(r)$ have the form $\xi(a) = 0$. By introducing the auxiliary vector [1]

$$\eta(r) = \left\| \begin{matrix} p(r) \\ q(r) \end{matrix} \right\|$$

which contains the two functions $p(r)$ and $q(r)$ related to the functions $u_2(r)$ and $w_2(r)$, we eliminate the second derivative from (3.10) and we therefore obtain an equation for the extended vector

$$r\mathbf{y}' = A\mathbf{y} + \alpha B\mathbf{y} \tag{3.11}$$

with the following boundary conditions

$$M\mathbf{y}(a) = 0$$

Here

$$\mathbf{y}(r) = \left\| \begin{matrix} u_2(r) \\ w_2(r) \\ p(r) \\ q(r) \end{matrix} \right\|, \quad A = \left\| \begin{matrix} P_0 & 0 \\ R_0 & S_0 \end{matrix} \right\|, \quad B = \left\| \begin{matrix} P_1 & Q_1 \\ R_1 & S_1 \end{matrix} \right\|, \quad M = \left\| \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \right\|$$

$$P_0 = \left\| \begin{matrix} \psi_1(r,\omega) & 0 \\ 0 & \psi_2(r,\omega) \end{matrix} \right\|, \quad R_0 = \left\| \begin{matrix} 0 & \psi_4(r,\omega) \\ \dfrac{1-k}{ak} r\psi_1(r,\omega) & 0 \end{matrix} \right\|$$

$$S_0 = \left\| \begin{matrix} -\psi_1(r,\omega) & 0 \\ 0 & -\psi_2(r,\omega) \end{matrix} \right\|$$

$$P_1 = \left\| \begin{matrix} 0 & \psi_3(r,\omega) \\ 0 & 0 \end{matrix} \right\|, \quad Q_1 = \left\| \begin{matrix} \dfrac{1}{k} & 0 \\ 0 & k \end{matrix} \right\|, \quad R_1 = \left\| \begin{matrix} -\dfrac{r^2}{a^2} & 0 \\ 0 & \dfrac{1-k}{ak} r\psi_3(r,\omega) - \dfrac{r^2}{a^2} \end{matrix} \right\|$$

$$S_1 = \left\| \begin{matrix} 0 & \dfrac{k(1-k)}{a} r - k^2\psi_3(r,\omega) \\ \dfrac{1-k}{ak^2} r & 0 \end{matrix} \right\|$$

The functions $\psi_1(r,\omega)$, $\psi_2(r,\omega)$ and $\psi_3(r,\omega)$ are solutions of the ordinary differential equations

$$\frac{d\psi_1}{dr} + \frac{1}{r}\psi_1^2 + \frac{\omega^2}{v_1^2} r = 0, \qquad \psi_1(0,\omega_k) = 0$$

$$\frac{d\psi_2}{dr} + \frac{1}{r}\psi_2^2 - \frac{1}{r} + \frac{\omega^2}{v_2^2} r = 0, \qquad \psi_2(0,\omega_k) = 1$$

$$\frac{d\psi_3}{dr} + \frac{\psi_3}{r}(\psi_1 + \psi_2) = \frac{2(1-k)}{ak}, \qquad \psi_3(0,\omega_k) = 0$$

The function $\psi_4(r,\omega)$ is determined by the following formula:

$$\psi_4(r,\omega) = \frac{1-k}{a} r [\psi_2(r,\omega) + 1] - \frac{2(1-k)}{a} r$$

The auxiliary vector $\eta(r)$ depends on $\xi(r)$, α, ω

$$\eta(r) = \frac{r}{\alpha} Q_1^{-1}\xi'(r) - \frac{1}{\alpha} Q_1^{-1}P_0\xi(r) - Q_1^{-1}P_1\xi(r)$$

The boundary value problem (3.11) is self-adjoint [4]. The matrix of the nondegenerate transformation $\mathbf{z} = T\mathbf{y}$ has the following form:

$$T(r) = \frac{1}{r} T_1 = \frac{1}{r} \left\| \begin{matrix} 0 & \chi(r,\omega) & \frac{1}{k} & 0 \\ -\chi(r,\omega) & 0 & 0 & -1 \\ -\frac{1}{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix} \right\|$$

$$\chi(r, \omega) = \psi_3(r, \omega) - (1-k)\, r/ak$$

It is easy to see that the following orthogonality condition

$$(\alpha_m - \alpha_n) \int_0^a \mathbf{y}_m{}^T Q \mathbf{y}_n dr = [\mathbf{y}_m{}^T T_1{}^T \mathbf{y}_n]_0{}^a = 0 \qquad (3.12)$$

is valid for the vectors y_m and y_n corresponding to the two distinct eigenvalues α_m and α_n of the parameter α. Here $\mathbf{y}_m{}^T$ is the transpose of the vector $y_m(r)$ corresponding to the eigenvalue α_m and Q is the nondegenerate matrix

$$B^T T = T^T B = Q = \left\| \begin{matrix} \frac{r}{ka^2} & 0 & 0 & 0 \\ 0 & \frac{\chi}{r}\psi_3 + \frac{(1-k)\psi_3}{ak} - \frac{r}{a^2} & \frac{1}{rk}\psi_3 & 0 \\ 0 & \frac{1}{rk}\psi_3 & \frac{1}{rk^2} & 0 \\ 0 & 0 & 0 & -\frac{k}{r} \end{matrix} \right\|$$

Using the orthogonality condition (3.12) we can determine the coefficients of the expansion of the arbitrary vector $\mathbf{y}_0(r)$ in a series in the vectors $\mathbf{y}_n(r)$ in the segment $[0, a]$. Let

$$\mathbf{y}_0(r) = \sum_{n=1}^{\infty} d_n \mathbf{y}_n(r)$$

Then there follows from (3.12)

$$d_n = \frac{1}{F_n} \int_0^a \mathbf{y}_n{}^T Q \mathbf{y}_0 dr \qquad (3.13)$$

On the basis of (3.2) and (3.7), F_n is here determined by the following formula:

$$F_n = \int_0^a \mathbf{y}_n{}^T Q \mathbf{y}_n dr = \frac{a}{\alpha_n} u_{2n}'(a) \left[\frac{\partial}{\partial\alpha} u_2(a, \alpha)\right]_{\alpha=\alpha_n} - \qquad (3.14)$$

$$\frac{a}{k\alpha_n} w_{2n}'(a) \left[\frac{\partial}{\partial\alpha} w_2(a, \alpha)\right]_{\alpha=\alpha_n}$$

Therefore, the constants d_n in (3.9) can be determined by the formulas (3.13) and (3.14) by interoducing the auxiliary vector $\eta(r)$.

4. As an illustration, let us consider the deformation of a semi-infinite cylinder with zero displacements on the curved surface, while on the endface $z = 0$ they are

$$u_2(r, 0, t) = r(1 - r/a)\, e^{i\omega t} \qquad (4.1)$$

$$w_2(r, 0, t) = 0$$

From (4.1) and (3.9) we obtain

$$\sum_{n=1}^{\infty} d_n u_{2n}(r) = r\left(1 - \frac{r}{a}\right) \tag{4.2}$$

Assuming $\eta_0(r) \equiv 0$, we have

$$\int_0^a \mathbf{y}_n{}^T Q \mathbf{y}_0 dr = \int_0^a \frac{r^2}{ka^2}\left(1 - \frac{r}{a}\right) u_{2n}(r)\, dr \tag{4.3}$$

Substituting the expression for $u_{2n}(r)$ from (3.6) into (4.3) and integrating, we obtain

$$\int_0^a \mathbf{y}_n{}^T Q \mathbf{y}_0 dr = \frac{\alpha_n V_1{}^2}{k(1-k)a^3\omega^2}\left[\frac{4}{\delta_{2n}^2} j_{12} - \frac{a}{\delta_{2n}} j_{02} - \frac{1}{\delta_{2n}} \int_0^a J_0(\delta_{2n} r)\, dr - \right.$$
$$\left. \frac{\alpha_n{}^2 j_{12}}{a^2 \delta_{1n} j_{11}}\left(\frac{4}{\delta_{1n}^3} j_{11} - \frac{a}{\delta_{1n}^2} j_{01} - \frac{1}{\delta_{1n}^2}\int_0^a J_0(\delta_{1n} r)\, dr\right)\right] \tag{4.4}$$

Furthermore, using (3.14), we determine F_n

$$F_n = \frac{j_{12}}{a(1-k)^2}\left[\frac{2}{\delta_{2n}} j_{02} + \frac{\alpha_n{}^2 k}{a\delta_{1n}^2} j_{12} - \frac{\alpha_n{}^2}{a\delta_{1n}\delta_{2n}} \frac{j_{01} j_{02}}{j_{11}}\right] \tag{4.5}$$

where $j_{ik} = J_i(\delta_{kn} a)$ is taken in (4.4) and (4.5).

It can be seen that for $\omega_k = 0$ we have

$$\psi_1(r, 0) = \psi_4(r, 0) = \chi(r, 0) = 0, \quad \psi_2(r, 0) = 1, \quad \psi_3(r, 0) = r(1-k)/ak$$

and we obtain the solution of the statics problem for a cylinder by passing to the limit $\omega \to 0$ in (3.2) or in (4.4) and (4.5).

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